Short time expansion for first passage distributions

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The probability that a stationary correlated Gaussian process does not cross zero in an interval of length t is computed to fifth order in t by a short time expansion. The difficulties inherent in finding higher order terms are discussed. The expansion is tested by simulations and comparison to some approximate results. [S1063-651X(97)00209-2]

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I. INTRODUCTION

The theory of Gaussian random processes is extremely well understood. It is used extensively in many branches of physics as a model for stochastic processes and as a starting point for field theories. A zero average Gaussian process is uniquely defined through its correlation function $C(t_1,t_2)$ and any statistical property of the process can be written as a functional of C. It is therefore somewhat surprising that the problem of the first passage time distribution for a general Gaussian process is unsolved.

A first passage time problem is the time after which one of the coordinates of a trajectory of a dynamical system will arrive for the first time at a specified position. If a dynamical system is being perturbed by a stochastic term, it is natural to consider the probability that the coordinate returns to its initial value after a time t [1]. This distribution is called the first return time distribution (FRTD) of the coordinate.

This kind of problem has important applications in chemistry and signal transmission [1] and has been discovered recently to be related to the kinetic behavior of the coarsening of phase ordering systems [2]. In addition, some problems in interface depinning can be cast as first passage time problems [3,4].

This paper concentrates on stationary Gaussian processes $X(t)$ with correlation functions that decay at least exponentially for long times and that possess an expansion in powers of τ^2:

$$\langle X(t)X(t+\tau)\rangle = C(\tau) = 1 + c_2\tau^2 + c_4\tau^4 + c_6\tau^6 + \cdots . \tag{1}$$

The question is what is the probability $F(t)$ that an interval of size t is free of a zero crossing [that is $X(\tau)>0$ for $0<\tau<t$].

This question has a long history, and has been approached in different ways. If the Gaussian process can be cast as a finite dimensional Fokker-Planck (FP) equation, the problem becomes one of finding the solution of the FP equation with a set of nontrivial boundary conditions [5]. If one is interested in long time asymptotics, one must extract the lowest lying eigenvalue of the FP operator with these boundary conditions. Not only is this a formidable task [6,7], but it precludes discussion of cases where the Gaussian process cannot be written as a stochastic differential equation of the type

$$a_0X + a_1\partial_t X + \cdots + a_n\partial_t^n X = \eta(t), \tag{2}$$

with a finite number of derivatives. In fact, the case in Eq. (1) is an example of a process that cannot be written [8] in the form of Eq. (2).

If one is interested in the asymptotic behavior of $F(t)$ and X is close in some sense to the Langevin process

$$X + \partial_t X = \eta \tag{3}$$

then one can use the perturbation expansion of Refs. [9,10]. There is also an approximate nonperturbative method that works for exponentially decaying correlation functions with a finite average zero crossing density [implying that $C''(0)$ exists]. This approximation is known to give good results for the whole distribution [11]. Finally, a method due to Rice [1] is to write down formally a series whose terms are related to the probability of having at least k intersections in the interval. The terms in the series consist of Gaussian integrals with complicated boundaries and difficult quadratic forms in the exponent. To our knowledge, these terms have not been computed beyond the first term. The method used in this paper is related to this idea, but we believe it is simpler to implement.

The main result of this paper is Eq. (44) for the probability that a segment of size t does not contain a zero crossing.

The paper is organized as follows. In Sec. II we describe the short time expansion and compute it to order t^5. In Sec. III we compare these exact results to known approximate results and to simulations. Section IV concludes and discusses some methods to extend these results.

II. SHORT TIME EXPANSION

A. General considerations

The problem of whether an interval is crossing-free is the same as whether a sample path X is all above or all below zero in the interval. Let us assume without limiting the generality that we want to know if the path is *below* zero. It is sufficient to check if the *maximum* along the path is negative, hence

$$\frac{1}{2}F(t) = \mathrm{Prob}(X_{\max}<0), \tag{4}$$

with $X_{\max} = \max_{\tau\in[0,t]} X(t)$. The factor $\frac{1}{2}$ takes into account the symmetric case where the path is above zero (in which case we ask for the probability that $X_{\min}>0$). We note in passing that the problem of first passage is related to the problem of finding the extremum of a random set (the path). This kind of ''ground state'' problem has been treated extensively in the theory of disordered systems [12], and the first passage time problem could perhaps be discussed within this framework [13]. In any case, a knowledge of

$$P(X_{\max})=\langle \delta(X_{\max}-\max_{\tau\in[0,t]} X(t))\rangle \tag{5}$$

yields $F(t)$ via an integration

$$\frac{1}{2}F(t)=\int_{-\infty}^{0} dXP(X). \tag{6}$$

B. The expansion

We now make the following observation: For a very short interval, the random process X can be approximated by a few terms in its Taylor series,

$$X(\tau)=\psi_0+\tau\psi_1+\tau^2\psi_2+\tau^3\psi_3+\cdots . \tag{7}$$

The form of Eq. (1) ensures that the coefficients $\psi_n=(1/n!)\partial_t^n X(t)|_{t=0}$ are well defined Gaussian variables. Their (cross) correlations can be deduced using Eq. (1) and we will discuss them later. For the moment, we need to find the probability that a polynomial with random coefficients does not vanish in the interval $[0,t]$. If we assume without limiting the generality that $\psi_0<0$ this is equivalent to finding the probability that $X_{\max}=\max_{\tau\epsilon[0,t]}X(t)$ is negative.

We want to find $X_{\max}$ in powers of t. Formally we write

$$X_{\max}=\psi_0+tz_1(\psi_1,\psi_2,\psi_3,\ldots)+t^2z_2(\psi_1,\psi_2,\psi_3,\ldots)+\cdots , \tag{8}$$

where there is no t dependence in the z_n functions (we also define $X=\psi_0+Y$ for convenience). The distribution $P(X_{\max})$ is defined as

$$P(X_{\max})=\langle \delta(X_{\max}-\psi_0-Y_{\max})\rangle,$$

which may be expanded in $Y_{\max}$,

$$P(X_{\max})=\langle \delta(X_{\max}-\psi_0)\rangle-\partial_{X_{\max}}\langle Y_{\max}\delta(X_{\max}-\psi_0)\rangle+\partial^2_{X_{\max}}\left\langle \frac{Y^2_{\max}}{2}\delta(X_{\max}-\psi_0)\right\rangle+\cdots . \tag{9}$$

We see that we need to develop a method to compute moments of $Y_{\max}$ in powers of t.

We would like to proceed perturbatively. We will ignore the possibility that $\psi_n\gg1$ because the ψ's are Gaussian random numbers with variances of order one, so the probability of this event is very small. We will only consider cases for which ψ's are order one or smaller.

If we have $\psi_1\gg t$ (an event that happens with a probability that approaches 1 for small t), we can approximate Y_τ by a straight line and treat the rest of the contributions as a small perturbation. This means that $Y_{\max}$ occurs at one of the end points $\tau=0,t$. The value of $Y_{\max}$ is of order t in this case (at least if the slope is positive) so this case will contribute to z_1 and higher terms in the expansion.

If, instead, $\psi_1\sim t$ but $\psi_2\sim1$ we must treat Y_τ as a parabola plus corrections that may be treated perturbatively. It is impossible to reach the parabola maximum perturbatively from the linear approximation since we only consider the end points. Notice that the amplitude of the maximum of such a parabola is of order t^2 and the probability of this event (e.g., $\psi_1\sim t$) is of order t so we expect this event to contribute to the z_3 term (and higher) in the expansion.

A similar consideration shows that the case $\psi_1\sim t^2$ and $\psi_2\sim t$ occurs with probability t^3 and with amplitude t^3 so it appears only in the z_6 term. This means that our expansion is accessible up to order t^5 by expanding only around the extremum of a parabola.

We see that the difficulty in generating a short time expansion is due to the need to find the extrema of polynomials of higher and higher order. This is an intractable problem for high order polynomials and becomes quite complicated even for a third order polynomial. The program is to consider only the case of corrections around a parabolic peak inside the interval. This will generate an expansion that is correct to order t^5.

C. The parabolic peak

We now consider the case where a single maximum occurs inside the interval. As noted above, for the maximum to occur inside the interval we must have $\psi_1\sim t$. To keep track of the expansion we define

$$\phi_1=t\psi_1, \tag{10}$$

$$u=\frac{\tau}{t}, \tag{11}$$

so that

$$Y=t^2(u\phi_1+u^2\psi_2+tu^3\psi_3+t^2u^4\psi_4+t^3u^5\psi_5+\cdots). \tag{12}$$

The extremum of such a parabola occurs at

$$u^*=u_0^*+tu_1^*+t^2u_2^*+t^3u_3^*+\cdots, \tag{13}$$

with

$$u_0^*=-\frac{\phi_1}{2\psi_2},$$

$$u_1^*=-\frac{3\phi_1^2\psi_3}{8\psi_2^3},$$

$$u_2^*=-\frac{\phi_1^3}{16\psi_2^5}(9\psi_3^2-4\psi_2\psi_4),$$

$$u_3^*=-\frac{5\phi_1^4}{128\psi_2^7}(27\psi_3^3-24\psi_2\psi_3\psi_4+4\psi_2^2\psi_5), \tag{14}$$

which gives

$$Y_{tu*}=t^2\left(-\frac{\phi_1^2}{4\psi_2}-t\frac{\phi_1^3\psi_3}{8\psi_2^3}+t^2\frac{\phi_1^4}{64\psi_2^5}(4\psi_2\psi_4-9\psi_3^2)\right.$$
$$\left.-t^3\frac{\phi_1^5}{128\psi_2^7}(27\psi_3^3-24\psi_2\psi_3\psi_4+4\psi_2^2\psi_5)\right). \quad (15)$$

In order for the extremum to be in the right range we demand that

$$0<u^*<1. \quad (16)$$

In addition we want the extremum to be a maximum so we demand that

$$\psi_2<0. \quad (17)$$

Clearly the expansion for u^* and Y^* can be continued to higher order. In order to be systematic in t it is necessary to include terms due to the third order polynomial peak as well. However, even without considering nonperturbative terms due to the cubic, higher orders for the quadratic alone can still be useful since they produce an upper bound on F. This point is discussed in the discussion.

D. Expansion for $Y_{\max}$

To order t^5 we have

$$Y_{\max}=\theta(Y_t)Y_t+\Omega_{0,1}(u^*)\theta(-\psi_2)[Y_{tu*}-\theta(Y_t)Y_t], \quad (18)$$

which encodes the fact that the maximum is either at one of the interval ends or in the middle of the interval if a quadratic maximum is in the interval. We have defined the function

$$\Omega_{a,b}(z)=1 \quad \text{for} \quad a<z<b \quad \text{and} \quad 0 \quad \text{elsewhere}. \quad (19)$$

Since we need the moments of $Y_{\max}$ we note that

$$\begin{aligned}(Y_{\max})^n&=\{\theta(Y_t)[1-\theta(-\psi_2)\Omega_{0,1}(u^*)]Y_t\\&\quad+\Omega_{0,1}(u^*)\theta(-\psi_2)Y_{tu*}\}^n\\&=\theta(Y_t)[1-\theta(-\psi_2)\Omega_{0,1}(u^*)](Y_t)^n+\Omega_{0,1}(u^*)\\&\quad\times\theta(-\psi_2)(Y_{tu*})^n, \end{aligned} \quad (20)$$

due to the projection properties of θ and Ω.

Thus we define

$$I_n=\theta(Y_t)(Y_t)^n \quad (21)$$

and

$$J_n=\Omega_{0,1}(u^*)\theta(-\psi_2)[(Y_{tu*})^n-\theta(Y_t)(Y_t)^n], \quad (22)$$

with $Y_{\max}^n=I_n+J_n$. Since we are interested in computing the averages in Eq. (9), we need to know how to integrate $Y_{\max}^n$ over functions of ψ_k. The main problem is the θ and Ω constraint functions. We can treat this systematically for small t. Let us consider the integrals of I_n, J_n over test functions of ψ_1.

$$\int I_nA(\psi_1)d\psi_1=\int_0^\infty (Y_t)^nA(\psi_1)d\psi_1-\int_0^\beta (Y_t)^nA(\psi_1)d\psi_1,$$

with $\beta=-(t\psi_2+t^2\psi_3+t^3\psi_4+t^4\psi_5+\cdots)$. Both integrals can be easily expanded in powers of t (note that β is small). This is just the linear approximation. Now we want to consider the J_n terms. For this we have to expand the constraint $0<u^*<1$. Recalling Eqs. (13) and (14), we see that the lower bound implies $\phi_1>0$ (since $\psi_2<0$); for the upper bound we can solve for $\phi_1^>$ such that $u^*(\phi_1^>)=1$. An expansion in t yields

$$\phi_1^>=-2\psi_2-t(3\psi_3)-t^2(4\psi_4)-t^3(5\psi_5) \quad (23)$$

so we have the constraint

$$0<\psi_1<-[2t\psi_2+t^2(3\psi_3)+t^3(4\psi_4)+t^4(5\psi_5)+\cdots]. \quad (24)$$

In order to evaluate

$$J_n=\Omega_{0,1}(u^*)\theta(-\psi_2)[(Y_{tu*})^n-\theta(Y_t)(Y_t)^n]$$

we gauge its effect on a test function $A(\psi_1)$,

$$\begin{aligned}\int A(\psi_1)J_nd\psi_1&=\int A(\psi_1)\Omega_{0,1}(u^*)\theta(-\psi_2)\\&\quad\times[(Y_{tu*})^n-\theta(Y_t)(Y_t)^n]d\psi_1\\&=\theta(-\psi_2)\int_0^\alpha (Y_{tu*})^nA(\psi_1)d\psi_1\\&\quad-\theta(-\psi_2)\int_\beta^\alpha (Y_t)^nA(\psi_1)d\psi_1,\end{aligned}$$

with $\alpha=-[2t\psi_2+t^2(3\psi_3)+t^3(4\psi_4)+t^4(5\psi_5)+\cdots]$.

Now an expansion in t is straightforward, leaving us with

$$\begin{aligned}Y_{\max}&=t[\psi_1\theta(\psi_1)]+t^2[\psi_2\theta(\psi_1)]\\&\quad+t^3\left[\psi_3\theta(\psi_1)+\psi_2^2\left(\frac{1}{2}+\frac{1}{6}\theta(-\psi_2)\right)\delta(\psi_1)\right]\\&\quad+t^4\left[\psi_4\theta(\psi_1)+\psi_2\psi_3\left(1+\frac{1}{2}\theta(-\psi_2)\right)\delta(\psi_1)\right.\\&\quad\left.+\psi_2^3\left(\frac{1}{6}+\frac{\theta(-\psi_2)}{6}\right)\delta'(\psi_1)\right]\\&\quad+t^5\left[\psi_5\theta(\psi_1)+\psi_3^2\left(\frac{1}{2}+\frac{2}{5}\theta(-\psi_2)\right)\delta(\psi_1)\right.\\&\quad+\psi_2\psi_4\left(1+\frac{3}{5}\theta(-\psi_2)\right)\delta(\psi_1)\\&\quad+\psi_2^2\psi_3\left(\frac{1}{2}+\frac{7}{10}\theta(-\psi_2)\right)\delta'(\psi_1)\end{aligned}$$

$$+\psi_2^4\left(\frac{1}{24}+\frac{11}{120}\theta(-\psi_2)\right)\delta''(\psi_1)\Bigg], \tag{25}$$

$$(Y_{\max})^2=t^2[\theta(\psi_1)\psi_1^2]+t^3[2\psi_1\psi_2\theta(\psi_1)] +t^4[(\psi_2^2+2\psi_1\psi_3)\theta(\psi_1)] +t^5\left\{(2\psi_2\psi_3+2\psi_1\psi_4)\theta(\psi_1) +\psi_2^3\left[\frac{1}{3}-\frac{2}{5}\theta(-\psi_2)\right]\delta(\psi_1)\right\}, \tag{26}$$

$$(Y_{\max})^3=t^3[\psi_1^3\theta(\psi_1)]+t^4[3\psi_1^2\psi_2\theta(\psi_1)] +t^5[(3\psi_1\psi_2^2+3\psi_1^2\psi_3)\theta(\psi_1)], \tag{27}$$

$$(Y_{\max})^4=t^4[\psi_1^4\theta(\psi_1)]+t^5[4\psi_1^3\psi_2\theta(\psi_1)], \tag{28}$$

$$(Y_{\max})^5=t^5[\psi_1^5\theta(\psi_1)]. \tag{29}$$

This result is quite general, and is independent of our assumption that ψ_k are Gaussian random variables. In the following section we shall assume that ψ_k are indeed random Gaussian variables with (cross) correlations that are implied by Eq. (1).

E. Averages

We will now proceed to compute averages of the type $\langle Y_{\max}^n\delta(X_{\max}-\psi_0)\rangle$. We write schematically

$$\psi_0=\eta_0, \tag{30}$$

$$\psi_1=\gamma_1^1\eta_1, \tag{31}$$

$$\psi_2=\gamma_0^2\eta_0+\gamma_2^2\eta_2, \tag{32}$$

$$\psi_3=\gamma_1^3\eta_1+\gamma_3^3\eta_3, \tag{33}$$

$$\psi_4=\gamma_0^4\eta_0+\gamma_2^4\eta_2+\gamma_4^4\eta_4, \tag{34}$$

$$\psi_5=\gamma_1^5\eta_1+\gamma_3^5\eta_3+\gamma_5^5\eta_5, \tag{35}$$

where $\langle\eta_i\eta_j\rangle=\delta_{ij}$ The values of γ_i^j in terms of c_k of the correlation function [Eq. (1)] will be given in Appendix A. It is useful to make the definitions

$$\langle\psi_2^n\psi_4^m\delta(X-\psi_0)\rangle=L_{n,m}, \tag{36}$$

$$\langle\psi_2^n\psi_4^m\theta(-\psi_2)\delta(X-\psi_0)\rangle=T_{n,m}, \tag{37}$$

whose explicit values are given in Appendix B. We find

$$\langle Y_{\max}\delta(X-\psi_0)\rangle=t\frac{\gamma_1^1}{\sqrt{2\pi}}L_{0,0}+t^2\frac{1}{2}L_{1,0} +t^3\left[\frac{\gamma_1^3}{\sqrt{2\pi}}L_{0,0}+\frac{1}{\gamma_1^1\sqrt{2\pi}}\left(\frac{L_{2,0}}{2}+\frac{T_{2,0}}{6}\right)\right] +t^4\left(\frac{1}{2}L_{0,1}\right)+t^5\left[\frac{\gamma_1^5}{\sqrt{2\pi}}L_{0,0} +\frac{(\gamma_3^3)^2}{\gamma_1^1\sqrt{2\pi}}\left(\frac{L_{0,0}}{2}+\frac{2T_{0,0}}{5}\right)+\frac{1}{\gamma_1^1\sqrt{2\pi}}\left(L_{1,1} +\frac{3T_{1,1}}{5}\right) -\left(\frac{L_{2,0}}{2}+\frac{7T_{2,0}}{10}\right)\frac{\gamma_1^3}{(\gamma_1^1)^2\sqrt{2\pi}} -\left(\frac{L_{4,0}}{24}+\frac{11T_{4,0}}{120}\right)\frac{1}{(\gamma_1^1)^3\sqrt{2\pi}}\right], \tag{38}$$

$$\langle Y_{\max}^2\delta(X-\psi_0)\rangle =t^2\frac{(\gamma_1^1)^2}{2}L_{0,0}(X)+t^3\frac{2\gamma_1^1}{\sqrt{2\pi}}L_{1,0}+t^4\left(\gamma_1^1\gamma_1^3L_{0,0}+\frac{1}{2}L_{2,0}\right) +t^5\left[\frac{2\gamma_1^3}{\sqrt{2\pi}}L_{1,0}+\frac{2\gamma_1^1}{\sqrt{2\pi}}L_{0,1}+\frac{1}{\gamma_1^1\sqrt{2\pi}}\left(\frac{L_{3,0}}{3}-\frac{2T_{3,0}}{5}\right)\right], \tag{39}$$

$$\langle Y_{\max}^3\delta(X-\psi_0)\rangle=t^3\frac{2(\gamma_1^1)^3}{\sqrt{2\pi}}L_{0,0}+t^4\frac{3(\gamma_1^1)^2}{2}L_{1,0} +t^5\left(3L_{2,0}\frac{\gamma_1^1}{\sqrt{2\pi}}+\frac{6(\gamma_1^1)^2\gamma_1^3}{\sqrt{2\pi}}L_{0,0}\right), \tag{40}$$

$$\langle Y_{\max}^4\delta(X-\psi_0)\rangle=t^4\frac{3(\gamma_1^1)^4}{2}L_{0,0}+t^5L_{1,0}\frac{8(\gamma_1^1)^3}{\sqrt{2\pi}}, \tag{41}$$

$$\langle Y_{\max}^5\delta(X-\psi_0)\rangle=t^5\frac{8(\gamma_1^1)^5}{\sqrt{2\pi}}L_{0,0}. \tag{42}$$

All the integrations and substitutions are best left to a computer algebra package (in this case MATHEMATICA).

We get that

$$\frac{1}{2}F(t)=\frac{1}{2}-\frac{\gamma_1^1}{2\pi}t+\frac{12\gamma_0^2(\gamma_1^1)^2+4(\gamma_1^1)^4-12\gamma_1^1\gamma_1^3-7(\gamma_2^2)^2}{24\gamma_1^1\pi}t^3+\left(\frac{-[(\gamma_0^2)^2\gamma_1^1]}{2\pi}+\frac{\gamma_0^4\gamma_1^1}{2\pi}-\frac{\gamma_0^2(\gamma_1^1)^3}{2\pi}-\frac{(\gamma_1^1)^5}{10\pi}+\frac{\gamma_0^2\gamma_1^3}{2\pi} +\frac{(\gamma_1^1)^2\gamma_1^3}{2\pi}+\frac{\gamma_0^2(\gamma_2^2)^2}{10\gamma_1^1\pi}+\frac{\gamma_1^1(\gamma_2^2)^2}{4\pi}-\frac{80(\gamma_1^1)^3\gamma_1^5-68\gamma_1^1\gamma_1^3(\gamma_2^2)^2-21(\gamma_2^2)^4+104(\gamma_1^1)^2\gamma_2^2\gamma_2^4+56(\gamma_1^1)^2(\gamma_3^3)^2}{160(\gamma_1^1)^3\pi}\right)t^5. \tag{43}$$

This expression becomes much simpler when written in terms of the correlation function's coefficients

$$F(t)=1-\frac{\sqrt{-2c_2}}{\pi}t+\frac{6c_4-c_2{}^2}{12\sqrt{-2c_2}\pi}t^3+\frac{43c_2{}^4-260c_2{}^2c_4-20c_4{}^2+80c_2c_6}{160\sqrt{-2c_2}c_2\pi}t^5+\text{ h.o.t.,} \tag{44}$$

where h.o.t. represents higher order terms. This is the main result of this work. The first order term is exactly the density of crossings. This is as expected since if the size of the intersection becomes infinitesimal, the probability of no crossing tends to 1 with a correction that is equal to the interval length t divided by the average distance between crossings $\langle l\rangle$. Hence we expect

$$F(t)=1-\frac{t}{\langle l\rangle}+\text{h.o.t.} \tag{45}$$

The cubic and quintic terms to our knowledge have not been computed before.

III. COMPARISONS

It is instructive to compare the exact expansion to the approximate results of Ref. [11]. In that work, the authors have an approximate closed formula for the Laplace transform of the probability $p(t)$ to find a spacing of size t between consecutive zero crossings.

$$\tilde{p}(\lambda)_{\text{approx}}=\frac{1-[\lambda^2\tilde{P}_+(\lambda)-\lambda]\tilde{p}'(0)}{1+[\lambda^2\tilde{P}_+(\lambda)-\lambda]\tilde{p}'(0)}, \tag{46}$$

where $\tilde{P}_+(\lambda)$ is the Laplace transform of $P_+(t)$, the probability that $X(0)X(t)>0$. In terms of the correlation function $C(t)$ one has

$$\tilde{P}_+(\lambda)=\frac{1}{\lambda}+\frac{1}{\pi\lambda}\int_0^\infty e^{-\lambda l}\frac{C'(l)}{\sqrt{1-C(l)^2}}dl \tag{47}$$

(see Ref. [11] for details). The expression for $p(t)$, Eq. (46), can be inverted for small t (that is, large λ) to yield

$$p(t)_{\text{approx}}=\frac{c_2^2-6c_4}{4c_2}t+\frac{-17c_2^4+108c_2^2c_4+156c_4^2-480c_2c_6}{96c_2^2}t^3+\text{h.o.t.} \tag{48}$$

The result of this expansion for $C(l)=1/\cosh(l/2)$ is

$$p(l)=\frac{l}{8}-\frac{3l^3}{128}+\frac{15l^5}{4096}+\cdots . \tag{49}$$

This expression is slightly different than the one given in Ref. [25] of [11] where Eq. (48) was computed for $C(l)=1/\cosh l$ but mistakenly attributed to $C(l)=1/\cosh(l/2)$ due to a misprint.

In order to compare to the exact expression, we recall that $F(t)$ is the probability that the interval $[0,t]$ is free of crossings, so it can be expressed as

$$F(t)=\frac{1}{\langle x\rangle}\int_t^\infty(x-t)p(x)dx, \tag{50}$$

with $\langle x\rangle=\int_0^\infty xp(x)dx=\pi/\sqrt{-2c_2}$ the average free interval size.

Thus

$$p(t)=\langle x\rangle F''(t)=\frac{c_2^2-6c_4}{4c_2}t+\frac{-43c_2^4+260c_2^2c_4+20c_4^2-80c_2c_6}{16c_2^2}t^3+\text{h.o.t.} \tag{51}$$

Notice that the first term of the approximate expression (48) agrees with the exact expression, but the next term is different. For example, for the correlation function given above, $C(l)=1/\cosh l/2$, we find that

$$p(t)=\frac{t}{8}+\frac{25t^3}{192}+\cdots . \tag{52}$$

In addition, we have performed some simulations to find the crossing distribution for several Gaussian processes. We considered a fifth order polynomial with Gaussian random correlated coefficients of the type (7) that are defined via Appendix A. Figure 1 plots the probability density to find the first zero crossing at a given time [which is just $-F'(t)$] and the expansion of $-F'(t)$ to order t^4. A very good agreement is found in the range $[0,0.8]$.

The increase in the simulated estimate of $F'(t)$ that appears after $t\sim0.9$ may appear puzzling since according to Eq. (50) $-F'(t)=(1/\langle x\rangle)\int_t^\infty(t)p(x)dx$ which is clearly monotonically decreasing. The resolution of this is to recall that the Gaussian process that is being simulated is only a polynomial approximation to a stationary process, accurate for short times, for larger times it is a nonstationary process and the probability $p(t)$ to find a spacing of size t between consecutive zero crossings is defined only for stationary processes. The correct interpretation of $-F'(t)$ is the probability density to find the first zero crossing beyond $t=0$ at a given time. This object need not be monotonic.

IV. DISCUSSION

A major shortcoming of the expansion derived in this work is the difficulty in calculating higher order terms. Following the method described above, one needs to integrate over the volume in the phase space of polynomial coefficients for which a polynomial never crosses zero in the interval $[0,t]$. The function to integrate over is a multivariate Gaussian with a quadratic form determined from the correlation function of the original problem. The shape of this

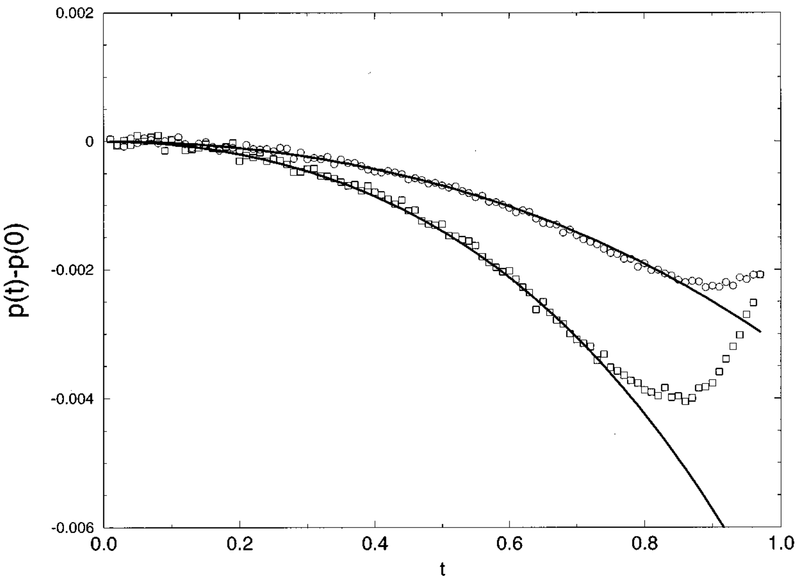

FIG. 1. The probability that the first zero crossing (after $t=0$) occurs at t versus t. The random processes used are polynomials of order τ^5 with coefficients $\psi_1, \ldots, \psi_5$ that are Gaussian variables of form (30) with γ's given by Eq. (A4). The correlation functions taken are $C(t)=1/\sqrt{\cosh(t/2)}$ (circles) and $C(t)=1/\cosh(t/2)$ (squares). The averages were performed over several tens of 10^6 of realizations. The solid lines are the results predicted by Eq. (44). The expansions are, respectively, $-F'(t)=0.056\,269\,8-0.002\,637\,65t^2-0.000\,563\,247t^4$ and $-F'(t)=0.079\,577\,5-0.004\,973\,75t^2-0.002\,590\,34t^4$. Each graph is shown with the zeroth order term subtracted.

region in phase space becomes increasingly complicated as higher orders in t are probed. This limits the feasible number of terms that can be obtained using this method. It would be difficult if not impossible to calculate enough terms to extract via Padé approximants or other methods the asymptotic exponential decay rate expected for $F(t)$ [11].

Although nominally the expansion is for short times, we can also view the expansion as one in *the number of extrema* inside the region. If there are no (local) extrema inside the region, it is sufficient to consider the end points. We then might use

$$Y_{\max}^{0-\mathrm{peak}}=\theta(Y_t)Y_t \tag{53}$$

to compute (either exactly or by perturbations) the value of $F^{0-\mathrm{peak}}(t)$. For one extremum, the parabolic approximation is sufficient etc. This is equivalent to performing a partial resummation and is an upper bound on $F(t)$. This can be seen by recalling that we are computing the probability that all the (local) extrema that we have considered are negative. Taking into account more extrema can only decrease this probability. A simple calculation gives

$$F^{0-\mathrm{peak}}(t)=2\frac{\sqrt{1-C(t)^2}}{2\pi}\int_{\pi}^{(7/4)\pi}\times\frac{d\theta}{\cos^2(\theta)+2[1-C(t)][\sin^2(\theta)+\sin(\theta)\cos(\theta)]} \tag{54}$$

or,

$$F^{0-\mathrm{peak}}(t)=1-\frac{2}{\pi}\arctan\left(\sqrt{\frac{1-C(t)}{1+C(t)}}\right), \tag{55}$$

whose large t asymptotics yields

$$F(t)<\frac{2}{\pi}C(t) \text{ for large } t. \tag{56}$$

This is a trivial observation: for all points to be positive, the two end points must be positive. A more sophisticated (but not rigorous) way of using this information was given in [11].

The single peak contribution cannot be resummed so easily, although a perturbation expansion can be produced quite straightforwardly to higher orders. Such a series should produce a more stringent bound on the $F(t)$ and is much simpler technically than the full expansion. In addition, an estimation of the asymptotic decay rate of this series can produce a better bound on the asymptotics of $F(t)$. Such an expansion is beyond the scope of this paper.

Finally, it would be interesting to try and apply this method to cases in which the Gaussian process is a surface (a random ''sheet''). Instead of asking about intervals in which the walk is positive, one asks about connected regions that are all positive (''islands''). This problem is known to exhibit some interesting features [14]. This method, if it could be extended, could shed some light on this problem.

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APPENDIX A

In this appendix we decompose a Gaussian process $X(\tau)$ with a correlation function

$$\langle X(t)X(t+\tau)\rangle=C(\tau)=1+c_2\tau^2+c_4\tau^4+c_6\tau^6+\cdots \tag{A1}$$

into

$$X(\tau)=\psi_0+\tau\psi_1+\tau^2\psi_2+\tau^3\psi_3+\cdots, \tag{A2}$$

where ψ_k's are correlated Gaussian random variables. We see that

$$\langle X(\epsilon)X(\tau)\rangle=\langle\psi_0^2\rangle+(\epsilon+\tau)\langle\psi_0\psi_1\rangle+[\tau\epsilon\langle\psi_1^2\rangle+(\epsilon^2+\tau^2)\langle\psi_0\psi_2\rangle]+\cdots . \tag{A3}$$

By expanding in ϵ and comparing to $C(\tau-\epsilon)$ we see that ψ_k have the form (30) with γ's given by

$$\gamma_1^1=\sqrt{-2c_2},$$

$$\gamma_0^2=c_2,$$

$$\gamma_2^2=\sqrt{6c_4-(\gamma_0^2)^2},$$

$$\gamma_1^3=-4c_4/\gamma_1^1,$$

$$\gamma_3^3=\sqrt{-20c_6-(\gamma_1^3)^2},$$

$$\gamma_0^4=c_4,$$

$$\gamma_2^4=(15c_6-\gamma_0^2\gamma_0^4)/\gamma_2^2,$$

$$\gamma_4^4=\sqrt{70c_8-(\gamma_2^4)^2-(\gamma_0^4)^2},$$

$$\gamma_1^5=-6c_6/\gamma_1^1,$$

$$\gamma_3^5=(-58c_8-(\gamma_1^5)^2)/\gamma_3^3,$$

$$\gamma_5^5=\sqrt{-252c_{10}}. \tag{A4}$$

Note that demanding real values for the γ's imposes a set of constraints on the coefficients of the correlation function. These ''realizability'' constraints are due to the fact that the eigenvalues of the correlation matrix must all be non-negative in order to ensure the normalizability of the Gaussian process.

APPENDIX B

The integrals

$$\langle\psi_2^n\psi_4^m\delta(X-\psi_0)\rangle=L_{n,m}, \tag{B1}$$

$$\langle\psi_2^n\psi_4^m\theta(-\psi_2)\,\delta(X-\psi_0)\rangle=T_{n,m} \tag{B2}$$

are given by

$$L_{n,m}=\langle(\gamma_0^2X+\gamma_2^2\eta_2)^n(\gamma_0^4X+\gamma_2^4\eta_2+\gamma_4^4\eta_4)^m\rangle P_0(X), \tag{B3}$$

$$T_{n,m}=\langle(\gamma_0^2X+\gamma_2^2\eta_2)^n(\gamma_0^4X+\gamma_2^4\eta_2+\gamma_4^4\eta_4)^m\times\theta(-\gamma_0^2x-\gamma_2^2\eta_2)\rangle P_0(x). \tag{B4}$$

For the values of n,m that are required for our expansion we have

$$L_{00}=e^{-X^2/2}/\sqrt{2\pi},$$

$$L_{10}=\gamma_0^2XL_{00}(X),$$

$$L_{20}=[(\gamma_0^2X)^2+(\gamma_2^2)^2]L_{00}(X),$$

$$L_{30}=[(\gamma_0^2X)^3+3\gamma_0^2X(\gamma_2^2)^2]L_{00}(X),$$

$$L_{40}=[(\gamma_0^2X)^4+6(\gamma_0^2X)^2(\gamma_2^2)^2+3(\gamma_2^2)^4]L_{00}(X),$$

$$L_{01}=\gamma_0^4XL_{00}(X),$$

$$L_{11}=(\gamma_0^2\gamma_0^4X^2+\gamma_2^2\gamma_2^4)L_{00}(X),$$

$$T_{00}=f0(X)L_{00}(X),$$

$$T_{20}=[(\gamma_0^2X)^2f0(X)+2\gamma_0^2\gamma_2^2Xf1(X)+(\gamma_2^2)^2f2(X)]L_{00}(X),$$

$$T_{30}=[(\gamma_0^2X)^3f0(X)+3(\gamma_0^2X)^2\gamma_2^2f1(X)+3\gamma_0^2X(\gamma_2^2)^2f2(X)+(\gamma_2^2)^3f3(X)]L_{00}(X),$$

$$T_{40}=[(\gamma_0^2X)^4f0(X)+4(\gamma_0^2X)^3\gamma_2^2f1(X)+6(\gamma_0^2X)^2(\gamma_2^2)^2f2(X)+4\gamma_0^2X(\gamma_2^2)^3f3(X)+(\gamma_2^2)^4f4(X)]L_{00}(X),$$

$$T_{11}=[\gamma_0^2\gamma_0^4X^2f0(X)+X(\gamma_0^2\gamma_2^4+\gamma_2^2\gamma_0^4)f1(X)+\gamma_2^2\gamma_2^4f2(X)]L_{00}(X), \tag{B5}$$

where we have defined the functions

$$fk(X)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\gamma_0^2x/\gamma_2^2}y^ke^{-y^2/2}dy. \tag{B6}$$

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tion function whose Fourier transform has a finite number of poles in ω. This means that the correlation function can be written as a finite sum $C(t)=\Sigma_{k=1,n}A_k e^{-\lambda_k|t|}$. This expression cannot have a Taylor expansion purely in terms of t^2. For this kind of correlation function, the expansion given in Eq. (7) will eventually break down due to the existence of terms of the type $\tau^m S_m(\tau)$ where $S_m(\tau)$ has a correlation $\langle S_m(\tau_1)S_m(\tau_2)\rangle \sim \min(\tau_1,\tau_2)$. The expansion method described here is expected to work until these terms start appearing in the expansion.

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